Additive-Output-Decomposition-Based Dynamic Inversion Tracking Control for a Class of Uncertain Linear Time-Invariant Systems

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Abstract—In this paper, the tracking control problem is investigated for a class of uncertain single-input single-output (SISO) linear time-invariant (LTI) systems. In a practical control system, uncertainties include parametric uncertainties, unmodeled dynamics, time delay and disturbances, which are too many and complicated to observe or know. By taking this into account, a new tracking controller design method is proposed, called the additive-output-decomposition-based dynamic inversion tracking control. The main idea is to lump all uncertainties together at the output by the proposed additive output decomposition. Then the transformed problem is handled by the dynamic inversion method. To demonstrate its effectiveness, the proposed tracking controller design is applied to two benchmark examples.

I. INTRODUCTION

In this paper, the tracking control problem for uncertain linear time-invariant (LTI) systems is investigated. Before introducing our main idea, some accepted control methods in the literature to handle uncertainties are briefly reviewed. A direct way is to estimate all of the unknown parameters, then compensate for them. In [1], a tracking problem for a linear system subject to unknown parameters and an unknown input delay was considered, where both the parameters and input delay were estimated by the proposed method. However, this method cannot handle nonparametric uncertainties such as unmodeled high-frequency gains. The second way is to design an adaptive controller to compensate for a part of unknown parameters but with robustness against other uncertainties. In [2], the Rohrs’ example and the two-cart example, which are tracking problems for uncertain linear systems subject respectively to unmodeled dynamics and time delay at the input, were revisited by $L_1$ adaptive control. In [3], the authors showed that their proposed method is robust against time delay at the input. Since each unknown parameter needs an integrator to estimate, for example see Equ. (5)-(7) in [3], an adaptive controller may require many integrators for an uncertain system with many unknown parameters. This will lead to a resulting closed-loop system with a reduced stability margin. In addition, the estimates may not approach the true parameters without persistently exciting signals, which are difficult to generate in practice especially when the number of unknown parameters is large [4, pp. 111-118]. A third way is to convert a tracking problem to a stabilization problem by the idea of internal model principle [5], if disturbances or desired trajectories are generated by an autonomous system. In [6], the problem of set point output tracking of an uncertain linear system with multiple delays in both the state and control vectors was considered. There also exist other methods to handle uncertainties. However, some of them such as high-gain feedback often cannot be applied to a practical system directly as they rely on a rapidly changing control signal to attenuate uncertainties and disturbance. These drawbacks of high-gain feedback solutions are that they may saturate the joint actuators or excite high-frequency modes.

For such a purpose, a new control scheme based on the additive output decomposition$^1$, called additive-output-decomposition-based dynamic inversion tracking control, is proposed. The proposed additive output decomposition is a new type of decomposition different from the lower-order subsystem decomposition. Concretely, taking the system $\dot{x}(t) = f(t, x), y = g(t, x), y \in \mathbb{R}^n$ for example, it is decomposed into two subsystems: $\dot{x}_1(t) = f_1(t, x_1, x_2), y_1 = y_1(t, x_1, x_2)$ and $\dot{x}_2(t) = f_2(t, x_1, x_2), y_2 = y_2(t, x_1, x_2)$ where $y_1 \in \mathbb{R}^{n_1}$ and $y_2 \in \mathbb{R}^{n_2}$, respectively. The lower-order subsystem decomposition with respect to output satisfies

$$n = n_1 + n_2 \text{ and } y = y_1 \oplus y_2.$$ 

By contrast, the proposed additive output decomposition satisfies

$$n = n_1 = n_2 \text{ and } y = y_1 + y_2.$$ 

By additive output decomposition, the original uncertain LTI system is decomposed into a determinate ‘primary’ system and a (derived) uncertain ‘secondary’ system, where the original output is viewed as arising from the outputs of the two resulting systems. By taking the output of the secondary system as a lumped disturbance, the original system is input-output equivalently viewed as a determinate ‘primary’ system subject to the lumped disturbance at the output. Based on the equivalent system, a tracking controller is designed by the dynamic inversion method for both reference tracking and lumped disturbance rejection. The design idea is straightforward and can handle both nonparametric uncertainties and many uncertain parameters together since uncertain parameters and disturbances are all lumped into one disturbance. Also, the tracking tasks are not limited to exogenous signals generated by an autonomous system. In

$^1$In this paper we have extended “additive decomposition” in [7] with respect to output rather than state. So, the term “additive decomposition” is replaced with the more descriptive term “additive output decomposition”.

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addition, the resulting controller has a simple control structure. To demonstrate the effectiveness, the proposed tracking controller design is applied to two benchmark examples.

II. PROBLEM FORMULATION AND DYNAMIC INVERSION METHOD

A. Problem Formulation

This paper considers a class of single-input-single-output (SISO) LTI systems specified in the form of transfer function:

\[ y(s) = G(s) u(s) + d(s). \]  

(1)

Here \( y(t) \in \mathbb{R} \) is the output, \( u(t) \in \mathbb{R} \) is the input control and \( d(t) \in \mathbb{R} \) is an unmeasurable but bounded disturbance, \( t \geq 0 \). The functions \( y(s), u(s), d(s) \) are the Laplace transforms of \( y(t), u(t), d(t) \), respectively; \( G(s) \) is a proper transfer function. It is assumed that the parameters of the transfer function \( G(s) \) are uncertain. The reference signal is denoted as \( r(t) \in \mathbb{R}, t \geq 0 \), which is known and bounded. Define the tracking error as follows:

\[ e(t) = r(t) - y(t). \]

The objective is to design \( u(t) \) such that \( e(t) \to 0 \) as \( t \to \infty \) with good tracking accuracy, i.e., tracking error is ultimately bounded by a small value. In the following, for convenience, we will drop the notation \( s \) or \( t \) except when necessary for clarity.

B. Dynamic Inversion Method

Suppose that \( G \) is known exactly and \( G^{-1} \) is physically realizable. The latter assumption has two-fold meaning: i) \( G \) is minimum phase, namely the zeros of \( G \) have negative real parts; ii) \( G \) is only a proper transfer function but not strictly proper, namely the order is equal to that of the numerator. Under the two assumptions above, the dynamic inversion tracking controller design is represented as follows:

\[ u = G^{-1}(r - \hat{d}) \]  

(2)

where \( \hat{d} \) is the disturbance estimate by the observer \( \hat{d} = y - Gu \). It is easy to see that \( \hat{d} \equiv d \). Substituting (2) into (1) results in

\[ y = GG^{-1}(r - \hat{d}) + d = r - \hat{d} + d = r \]

where \( \hat{d} \equiv d \) has been utilized. As a result, perfect tracking is achieved. However, the proposed controller (2) cannot be realizable, even if \( G \) is known exactly and \( G^{-1} \) is physically realizable. For example, let \( G = 1 \) for simplicity, which is a proper system but not strictly proper system. Then controller (2) is written as \( u = r - \hat{d} \), where \( \hat{d} \) is obtained by \( \hat{d} = y - u \). It cannot be realized, since at time \( t \), the controller \( u(t) \) needs \( \hat{d}(t) \), on the other hand the estimate \( \hat{d}(t) \) needs \( u(t) \) simultaneously. To avoid the dilemma, a modified way is proposed as follows:

\[ u = G^{-1}Q(r - \hat{d}) \]  

(3)

where \( Q \) is a low-pass filter making \( G^{-1}Q \) strictly proper. By employing the controller (3), the tracking error results in

\[ e = (1 - Q)(r - d). \]  

(4)

Since \( Q \) is a low-pass filter and the low-frequency range is often dominant in the signal \( r - d \), the tracking error will be attenuated by the transfer function \( 1 - Q \). For example, the reference \( r \) and the disturbance \( d \) are constant signals, namely \( r(s) = d(s) = \frac{1}{2}c \), where \( c \in \mathbb{R} \) is an unknown constant. Suppose \( Q(s) = \frac{1}{a+1} \), \( a \in \mathbb{R} \). Then according to (4) the tracking error is

\[ e(s) = \frac{c}{a+1} \]  

It is easy to see that the tracking error will approach zero. By incorporating the low-pass filter \( Q \), the proposed tracking controller can be realized and the transfer function \( G \) can be strictly proper as well. However, the dynamic inversion tracking controller design still requires \( G \) to be exact and the zeros of \( G \) to have negative real parts, especially the first requirement. In the next section, a new tracking controller design, called additive-output-decomposition-based dynamic inversion tracking controller design, is proposed to solve the tracking problem for uncertain systems (1). The proposed tracking controller design does not require \( G \) to be exact and the zeros of \( G \) to have negative real parts.

III. ADDITIVE-OUTPUT-DECOMPOSITION-BASED DYNAMIC INVERSION TRACKING CONTROLLER DESIGN AND ANALYSIS

First, additive output decomposition is proposed to make the paper self-contained. Then, the tracking controller is proposed based on additive output decomposition. After adding the designed controller into the considered system (1), the tracking performance is analyzed later.

A. Additive Output Decomposition

Let us consider a class of differential dynamic systems as follows:

\[ \dot{x} = f(t, x, u), x(0) = x_0 \]
\[ y = h(t, x) \]  

(5)

where \( f : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m_1 \to \mathbb{R}^n \), \( h : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}_2^m, x \in \mathbb{R}^n, u \in \mathbb{R}^m_1, y \in \mathbb{R}_2^m \) are the state, input and output, respectively. We first bring in a 'primary' system with output having the same dimension as (5), according to:

\[ \dot{x}_p = f_p(t, x_p, u_p), x_p(0) = x_{p,0} \]
\[ y_p = h_p(t, x_p) \]  

(6)

where \( f_p : [0, \infty) \times \mathbb{R}^{n_1} \times \mathbb{R}^{m_1} \to \mathbb{R}^{n_1} \), \( h_p : [0, \infty) \times \mathbb{R}^{n_1} \to \mathbb{R}^{m_2}, x_p \in \mathbb{R}^{n_1}, u_p \in \mathbb{R}^{m_1}, y_p \in \mathbb{R}^{m_2} \) are the state, input and output, respectively. Subtracting \( y_p \) from both sides of the term \( y = h(t, x) \) in (5) results in

\[ \dot{x} = f(t, x, u), x(0) = x_0 \]
\[ y - y_p = h(t, x) - y_p \]

Define new variables \( x_s \in \mathbb{R}^n \) and \( y_s \in \mathbb{R}^{m_2} \) as follows:

\[ x_s \triangleq x, y_s \triangleq y - y_p. \]  

(7)
Then we derive the following ‘secondary’ system:

\[
\begin{align*}
\dot{x}_s &= f(t, x_s, u), \quad x_s(0) = x_0 \\
y_s &= h(t, x_s) - h_p(t, x_p)
\end{align*}
\] (8)

where \(x_s \in \mathbb{R}^n, u \in \mathbb{R}^m, y_s \in \mathbb{R}^m\) are the state, input and output, respectively. From the definition (7), we have

\[
y(t) = y_p(t) + y_s(t), \quad t \geq 0.
\] (9)

Let us look at the system considered in this paper. Consider the LTI system (1) as the original system:

\[
y = Gu + d
\] (10)

where \(y \in \mathbb{R}\). We first bring in a primary system with output having the same dimension as (10), according to:

\[
y_p = G_p u_p
\] (11)

where \(y_p \in \mathbb{R}\) and \(G_p\) is called the ‘primary’ transfer function. From the original system (10) and the primary system (11) we derive the following secondary system by the rule (8):

\[
y_s = G u + d - G_p u_p
\] (12)

where \(y_s \equiv y - y_p\).

Remark 1. Unlike the additive decomposition (additive state decomposition) proposed in [7], [8], the state dimensions of the primary system and the secondary system here can be different.

B. Controller Design

Reformulate the resulting primary system (11) and secondary system (12) as follows:

\[
y_p = G_p u_p, \quad y = y_p + y_s.
\]

In the following, choose \(u_p = u\) and denote \(y_s = d_l\). Then

\[
y_p = G_p u, \quad y = y_p + d_l
\] (13)

where \(d_l = (G - G_p) u + d\) is called the lumped disturbance. Unlike the original disturbance \(d\) in (1), the lumped disturbance \(d_l\) includes uncertainties, disturbance and input. Fortunately, since \(G_p u\) and the output \(y\) are known, the lumped disturbance \(d_l\) can be observed by

\[
\hat{d}_l = y - G_p u.
\]

It is easy to see that \(\hat{d}_l \equiv d_l\). The input and output of systems (1) and (13) are the same. So, the system (13) is an input-output equivalent system of (1). So far, by the additive output decomposition, an uncertain system has been transformed into a determinate system but subject to a lumped disturbance, which is shown in Fig. 1. For the system (13), the primary transfer function \(G_p\) will be designed to be minimum phase to satisfy the requirement of dynamic inversion method. According to (3), the tracking controller for (13) is designed as follows:

\[
u = G_p^{-1} Q \left(r - \hat{d}_l\right).
\] (14)

As seen above, the proposed tracking controller (14) is straightforward with a simple control structure. Moreover, it is not required that \(G\) be exact and invertible. Instead, it only needs to satisfy some conditions depending on the uncertain transfer function \(G\) (they will be given in the following section). The design mentioned above is called additive-output-decomposition-based dynamic inversion tracking controller design.

C. Performance Analysis

Unlike the disturbance \(d\), the lumped disturbance \(d_l\) involves the input \(u\). So, an inappropriate \(G_p\) may cause instability of the resulting closed-loop system. Next, some conditions are given to ensure that the control input \(u\) is bounded.

**Theorem 1.** Let the tracking controller \(u\) for (1) be designed as in (14). Suppose i) \(G_p\) is minimum phase and \(Q\) is stable, ii) \((Q - (1 - GG_p^{-1}))^{-1}\) is stable, and iii) \(r - d\) is bounded. Then \(u\) is bounded.

**Proof.** Since \(d_l = d_l = (G - G_p) u + d\), the controller (14) is written as follows:

\[
u = G_p^{-1} Q (r - (G - G_p) u - d).
\]

Rearranging the controller above results in

\[
u = (G_p - Q (G_p - G))^{-1} Q (r - d)
\]

\[
= G_p^{-1} (1 - Q (1 - GG_p^{-1}))^{-1} Q (r - d).
\] (15)

Since \(G_p\) is minimum phase, \(G_p^{-1}\) is exponentially stable. Moreover, since the transfer function \((1 - Q (1 - GG_p^{-1}))^{-1}\) is stable, \(G_p^{-1} (1 - Q (1 - GG_p^{-1}))^{-1} Q\) is stable. In addition, since \(r - d\) is bounded by the assumption, the control input \(u\) is bounded according to (15).

By the small gain theorem [9, pp. 96-98], several conditions can be derived to ensure the stability of \((Q - (1 - GG_p^{-1}))^{-1}\).

**Theorem 2.** Let the tracking controller \(u\) for (1) be designed as in (14). Suppose i) \(G_p\) is minimum phase, and
Q, G are stable, ii)
\[ \sup_{\omega} \left| (1 - GG^{-1}_p) Q(j\omega) \right| < 1, \quad (16) \]
and iii) \( r - d \) is bounded. Then \( u \) is bounded.

**Proof.** Since \( G_p \) is minimum phase and \( Q, G \) are stable, the transfer functions \( 1 - GG^{-1}_p \) and \( Q \) are stable. By the small gain theorem [9, pp. 96-98], \( (1 - Q (1 - GG^{-1}_p))^{-1} \) is stable since (16) holds. It can conclude this proof by following Theorem 1. □

By employing the controller (15), the tracking error is represented as follows:
\[ e = GG^{-1}_p (1 - Q (1 - GG^{-1}_p))^{-1} Q (r - d) - (r - d) \]
\[ = \left( GG^{-1}_p (1 - Q (1 - GG^{-1}_p))^{-1} Q - 1 \right) (r - d). \]
The tracking error is further represented as follows:
\[ e = P (r - d) \quad (17) \]
where
\[ P = \left( 1 - (1 - GG^{-1}_p) Q \right)^{-1} (1 - Q). \]

**Definition 1** [10]. The \( L_1 \) gain of a stable proper SISO system is defined as \( \|G (s)\|_{L_1} = \int_0^\infty |g(t)| dt \), where \( g(t) \) is the impulse response of \( G (s) \).

**Theorem 3.** Suppose that the conditions of Theorem 1 or Theorem 2 hold. Then the tracking error is bounded by
\[ \|e\|_\infty \leq \|P\|_{L_1} \cdot \|r - d\|_\infty \quad (18) \]
where \( \|x\|_\infty \triangleq \sup_{t \geq 0} |x(t)| \). In particular, if \( r - d \) is constant and \( Q (0) = 1 \), then the tracking error \( e(t) \to 0 \) as \( t \to \infty \).

**Proof.** The proof for the first part is straightforward from (17). In particular, if the signal \( r - d \) is constant and \( Q (0) = 1 \), then \( L^{-1} [P_2 (r - d) (s)] \) is only an impulse, where \( L^{-1} \) denotes the inverse Laplace transformation. Since \( e = P_1 P_2 (r - d) \) and \( P_1 \) is stable, the tracking error \( e(t) \to 0 \) as \( t \to \infty \) by (17). □

**Remark 2.** In the controller design, we expect \( G_p \) to approximate \( G \) in the low frequency band as far as possible. Consequentially, the term \( GG^{-1}_p \approx 1 \) holds in the low frequency band. Readers can refer to [11],[12] for system identification or model approximation. Since \( Q \) is a low-pass filter, we have \( (1 - GG^{-1}_p) Q \approx 0 \). Furthermore, \( (1 - Q) (1 - (1 - GG^{-1}_p))^{-1} \approx 1 - Q \) holds. The tracking error is represented approximately as follows:
\[ e \approx (1 - Q) (r - d). \quad (19) \]
Since \( Q \) is a low-pass filter, \( 1 - Q \approx 0 \) in the low frequency band and \( 1 - Q \approx 1 \) in the high frequency band. It is well known that the low frequency band is often dominant in the signal \( r - d \). As a result, good tracking performance is achieved according to (19).

The difference between \( G \) and \( G_p \) can also be represented in the relative or multiplicative form
\[ G = (1 + \Delta) G_p \quad (20) \]
where \( \Delta \) denotes the modeling error relative to \( G_p \). Since \( (1 - GG^{-1}_p) Q = \Delta Q \), the proofs of the following two corollaries are straightforward from Theorems 2-3.

**Corollary 1.** Let the tracking controller \( u \) for (1) be designed as (14). Suppose i) \( G_p \) is minimum phase, and \( Q, G \) are stable, ii) (20) holds, and iii) \( \sup_{\omega} |\Delta (j\omega) Q (j\omega)| < 1 \). Then \( u \) is bounded.

**Corollary 2.** Suppose that the conditions of Corollary 1 hold. Then the tracking error is bounded by
\[ \|e\|_\infty \leq \left\| (1 - Q) (1 + \Delta Q)^{-1} \right\|_{L_1} \cdot \|r - d\|_\infty. \]

**Remark 3.** Since \( \Delta (j\omega) \approx 0 \) in low frequency band and \( Q \) is a low-pass filter with \( Q (j\omega) \approx 0 \) in high frequency band, \( \Delta (j\omega) Q (j\omega) \approx 0 \) for all frequencies.

**Remark 4.** The transfer function \( G_p \) can be obtained by the system identification method offline or online. It is only expected that \( G_p \) capture the main feature of \( G \), while \( G_p \) does not need to have the same poles and zeros as \( G \). According to this, the identified parameters can be reduced by choosing a low-order \( G_p \).

**IV. NUMERICAL EXAMPLES**

In order to illustrate the additive-output-decomposition-based dynamic inversion tracking controller design explicitly, we provide the following two examples.

**A. Rohrs’ Example**

Consider the Rohrs’ example system as \( G (s) = \frac{2}{s^2 + 229} \frac{229}{s^4 + 30s + 229} \) which is assumed unknown. Assume that \( G_o (s) = \frac{1 - T s}{1 + T s} \) is obtained from the input and output data of the Rohrs’ example system. The objective is to design \( u \) such that \( y \to r \) or with good tracking accuracy (tracking error is ultimately bounded by a small value), where the reference signal is \( r (s) = \frac{1}{1 + T s} y_d (s) \). Assume \( y_d (t) = 1 \) for \( t \geq 1 \), \( y_d (t) = \sin (t) \), \( t \geq 1 \). The Rohrs’ example system in [13] was proposed to demonstrate that conventional adaptive control algorithms developed at that time lose their robustness in the presence of unmodeled dynamics. In the following, the proposed additive-output-decomposition-based tracking controller will revisit this example.

Choose \( G_p = G_o \) and \( Q (s) = \frac{1 + T s}{1 + T s} \) and then plot \( |(1 - GG^{-1}_p) (j\omega)| \), \( |Q (j\omega)| \) and \( |(1 - GG^{-1}_p) Q (j\omega)| \) in Fig.2. As shown, \( 1 - GG^{-1}_p \) is a high-pass filter close to zero in low frequency band, which implies \( G_p \) approximated \( G \) well in the low frequency band. The resulting \( |(1 - GG^{-1}_p) Q (j\omega)| \) achieves maximum (\( < 1 \)) around \( \omega = 10 \) rad/sec. Since \( G_p \) is minimum phase, and \( Q, G \) are stable, the resulting closed-loop system is stable according to Theorem 2.

By using the proposed controller (14) with the parameters above, the tracking response is shown in Fig.3. As shown, the specified tracking performance is achieved approximately and the tracking error of step signal approaches zero. These results are consistent with Theorem 3.

**Remark 5.** How to obtain \( G_p \) by system identification is omitted here because of space limitation. If the input
and output data are obtained, then the readers can refer to the “system identification toolbox” of Matlab software to get the transfer function \( G_p \) or refer to [12] for model approximation. The filter is often chosen to be \( Q = \frac{1}{s^N + 1} \) for simplicity, where \( N \) is a positive integer. After \( G_p \) is determined, the parameter \( \epsilon \) can be adjusted online to achieve good tracking performance.

### B. Two-Cart Example

The two-cart mass-spring-damper example was originally proposed as a benchmark problem for robust control design [2],[14]. Next, we will revisit the two-cart example by the proposed control scheme. The two-cart system is shown in Fig. 4, which is used as the primary transfer function. The filter is chosen to be \( Q(s) = \frac{1}{s^{0.02} + 1} \).

The magnitude \( |Q(j\omega)| \) is shown in Fig. 5, whose variation is around \( -0.0369 \). The transfer function from the input \( v \) to \( y \) is denoted as \( F(s) \), whose magnitude is shown in Fig. 5. As shown, \( |F(j\omega)| \) varies greatly in the low frequency band. We find that it is difficult to approximate \( F(s) \) by a lower order transfer function in the low frequency band. So, a feedback controller is designed to stabilize the matrix \( A \) as: 

\[
F(s) = \frac{0.1s + 0.15}{s^4 + 0.1089s^3 + 0.2056s^2 + 0.5278s + 0.66}.
\]

The magnitude \( |G(j\omega)| \) is shown in Fig. 5, whose variation is smooth. Approximate the transfer function by using a simpler one as follows:

\[
G_p(s) = \frac{0.1089}{s^3 + 0.869s^2 + 0.3558s + 0.04784}
\]

which is used as the primary transfer function. The filter is chosen to be \( Q(s) = \frac{1}{s^{0.02} + 1} \).

\[
G(s) = \frac{0.1s + 0.15}{s^4 + 0.1089s^3 + 0.2056s^2 + 0.5278s + 0.66}
\]
In this paper, a new tracking controller design, called additive-output-decomposition-based dynamic inversion tracking controller design, is proposed to tracking problem for a class of uncertain SISO LTI systems. Our main contribution lies in the presentation of a new decomposition scheme, named additive output decomposition, which helps to lump all uncertainties into a disturbance at the output and then offers a way to deal with many and various uncertainties together. Based on it, the controller is designed by the dynamic inversion method. The design is straightforward and the resulting controller is with a simple structure. From the given two examples, the proposed tracking controller design is applicable to the tracking problem for uncertain systems.

**V. Conclusions**

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